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Non-Triviality of a Discrete Bak–Sneppen Evolution Model

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Consider the following evolution model, proposed in ref. 1 by Bak and Sneppen. Put N vertices on a circle, spaced evenly. Each vertex represents a certain species. We associate with each vertex a random variable, representing the "state" or "fitness" of the species, with values in $[0, 1]$. The dynamics proceeds as follows. Every discrete time step, we choose the vertex with minimal fitness, and assign to this vertex, and to its two neighbours, three new independent fitnesses with a uniform distribution on $[0, 1]$. A conjecture of physicists, based on simulations, is that in the stationary regime, the one-dimensional marginal distributions of the fitnesses converges, when $N \to \infty$, to a uniform distribution on $(f, 1)$, for some threshold $f < 1$. In this paper we consider a discrete version of this model, proposed in ref. 2. In this discrete version, the fitness of a vertex can be either 0 or 1. The system evolves according to the following rules. Each discrete time step, we choose an arbitrary vertex with fitness 0. If all the vertices have fitness 1, then we choose an arbitrary vertex with fitness 1. Then we update the fitnesses of this vertex and of its two neighbours by three new independent fitnesses, taking value 0 with probability $0 < q < 1$, and 1 with probability $p = 1 - q$. We show that if q is close enough to one, then the mean average fitness in the stationary regime is bounded away from 1, uniformly in the number of vertices. This is a small step in the direction of the conjecture mentioned above, and also settles a conjecture mentioned in ref. 2. Our proof is based on a reduction to a continuous time particle system.

KEY WORDS: Species; fitness; evolution; interacting particle system; self-organised criticality; coupling; contact process; stationary distribution.

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1. INTRODUCTION

The Bak Sneppen model, introduced in ref. 1, has received a lot of attention in the literature, see, for instance, refs. 3–10. In ref. 3, it is described how Bak and Sneppen were looking for a simple mathematical model which was supposed to exhibit evolutionary behaviour, and which was also supposed to fall into the class of processes showing self-organised critical behaviour. For physicists, self-organised critical behaviour refers to power law decay of temporal and spatial quantities, without fine-tuning of parameters. After many attempts, Bak and Sneppen arrived at the following process.

Think of a system with N species. These species are represented by N vertices on a circle, evenly spaced. Now each of these species is assigned a so called "fitness," and in this model, the fitness is a number between 0 and 1. The higher the fitness, the better chance of surviving the species has. The dynamics of evolution is modelled as follows. Every discrete time step, we choose the vertex with minimal fitness, and we think of the corresponding species as disappearing completely. This species is then replaced by a new one, with a fresh and independent fitness, uniformly distributed on $[0, 1]$. So far, the dynamics does not have any interaction between the species, and does not result in an interesting process. Interaction is introduced by also replacing the two neighbours of the vertex with lowest fitness by new species with independent fitnesses. This interaction represents co-evolution of related species: if a certain species becomes extinct, this has an effect on other species as well. The neighbour interaction makes the model very interesting from a mathematical point of view.

It is extremely simple to run this model on a computer. Simulations then suggest the following behaviour, for large N (see refs. 3 and 4 for simulation results). It appears that the one-dimensional marginals are uniform (in the limit for $N \to \infty$) on $(f, 1)$ for some f whose numerical value is supposed to be close to $2/3$. This threshold value f is the basis for self-organised critical behaviour, according to refs. 1, 3, and 4. Since in the limit there is no mass below f, one can look at so called avalanches of fitnesses below this threshold: start counting at the moment that there is one fitness below f and wait until all fitnesses are above f again. The random number of updates, for instance, counted this way, is suppose to follow a power law, and there is no fine-tuning of parameters.

It is a challenge to prove any of the above statements. Note that in order to prove power law behaviour, one should first prove the existence of the threshold f with the property that in the limit for $N \to \infty$, all one-dimensional marginals are concentrated on $(f, 1)$. Indeed, one can define avalanches corresponding to other thresholds as well, but it is not expected that these avalanches have power law behaviour. This is only expected (and observed) for the self-organised threshold f. Therefore, this should be the starting point of a rigorous mathematical analysis of the model.

Simple as the model may appear, it turns out to be very difficult to say anything at all about the limiting one-dimensional distributions. It is therefore natural to try to prove a similar result in a simpler model. In this light, we have chosen to study a discrete version of the model, which was proposed in ref. 2, and which can be described as follows. Fitnesses of species can now only be 0 or 1. The dynamics in this simpler model proceeds as follows: at every discrete time step, we choose an arbitrary vertex with fitness 0. If there is no such vertex, then we choose an arbitrary vertex with fitness 1. We update the fitnesses of this vertex, and of its two neighbours, by three new independent fitnesses, taking value 0 with probability $0 < q < 1$, and 1 with probability $p = 1 - q$. This process is called the BS process in this paper. We show that if q is close enough to one, then the mean average fitness in the stationary regime is bounded away from 1, uniformly in the number of vertices. This is a small step in the direction of the conjecture mentioned above, and answers a question which was posed in ref. 2.

It should be noted that this discrete version of the model does not show self-organised critical behaviour. Nevertheless, we think that understanding of the discrete model also increases our understanding of the original model, if not in a technical sense, certainly in a conceptual sense. Admittedly, the discrete version suggested here and in ref. 2 is only one out of many possible discrete versions, but we see no reason to complicate matters unnecessarily by choosing more complicated discrete versions. The reader will notice that the proof of our main result is already quite complicated.

In order to state our main result, here follows some notation. As before, the number of vertices is denoted by N, and we denote by $\eta_N(n)_i$ the state of the ith vertex after n updates of the process. We will prove the following result.

Theorem 1.1. If q is close enough to one, then there exists $c_q > 0$ such that for any $N \in \mathbb{N}$ and $i \in \{1,..., N\}$,

$$\lim_{n \to \infty} P(\eta_N(n)_i = 0) \geqslant c_q. \tag{1.1}$$

Note that $\lim_{n \to \infty} P(\eta_N(n)_i = 0)$ exists, because $\eta_N(n)$ is a finite state, irreducible and aperiodic Markov chain.

In the next section, we reduce the problem to a problem in a continuous time, monotone particle system. In this system we will be able to prove results uniformly in N, by exhibiting graphical representations and an infinite space version of the particle system.

2. REDUCTION TO A MONOTONE CONTINUOUS TIME PROCESS.

In this section we define a useful continuous time stochastic process $\xi(t)$, independent of N. We construct the process $\xi(t)$ via a graphical representation. The graphical representation GR is a random graph on the space-time diagram $\mathbb{Z}\times\mathbb{R}^+$. We define GR via a set of independent so called *bundles* $\{\Pi_k\}_{k\in\mathbb{Z}}$, where each bundle Π_k consists of eight independent Poisson processes on $\mathbb{R}$,

$$\Pi_k=\{\Pi_k^{000},\Pi_k^{001},\ldots,\Pi_k^{110},\Pi_k^{111}\},$$

with parameters $q^3/(1-q^3)$, $q^2p/(1-q^3)$,..., $p^2q/(1-q^3)$, $p^3/(1-q^3)$ respectively. (We use the factor $1/(1-q^3)$ to rescale time in a convenient way, as will become clear later.) For each process $\Pi_k^{\sigma_1,\sigma_2,\sigma_3}$ we perform the following procedure. At ith arrival $\tau_{k,i}^{\sigma_1,\sigma_2,\sigma_3}$ of $\Pi_k^{\sigma_1,\sigma_2,\sigma_3}$, $i\in\mathbb{Z}$, we draw arrows in $\mathbb{Z}\times\mathbb{R}$ from $(k,\tau_{k,i}^{\sigma_1,\sigma_2,\sigma_3})$ to $(k-1,\tau_{k,i}^{\sigma_1,\sigma_2,\sigma_3})$, iff $\sigma_1=0$, and from $(k,\tau_{k,i}^{\sigma_1,\sigma_2,\sigma_3})$ to $(k+1,\tau_{k,i}^{\sigma_1,\sigma_2,\sigma_3})$, iff $\sigma_3=0$. We draw a $*$ in $\mathbb{Z}\times\mathbb{R}$ at every $(k+j,\tau_{k,i}^{\sigma_1,\sigma_2,\sigma_3})$ with $\sigma_j=1$, $j=-1,0,1$. We say that (x,t_1) is *connected* to (x,t_2) by a *time segment*, if $t_1<t_2$ and there are no $*$'s on (x,t), $t\in[t_1,t_2)$. An *open path* $\gamma(t)|_{t_1}^{t_2}$ with *trace* (x_0,s_0),..., (x_n,s_n) is a map from $[t_1,t_2]$ to $\mathbb{Z}$ such that $t_1=s_0<\cdots<s_n=t_2$, $\gamma(t)=x_i$, for $t\in[s_i,s_{i+1})$, $0\leqslant i<n$, $\gamma(t_2)=x_n$, and every pair (x_i,s_i), (x_{i+1},s_{i+1}) is connected either by a time segment or by an arrow. For any finite $B\subset\mathbb{Z}$, $x,y\in B$, and $t_1\leqslant t_2\in\mathbb{R}$, write $(x,t_1)\rightsquigarrow(y,t_2)$ in $GR|_B$, if there exists an open path $\gamma(t)|_{t_1}^{t_2}$ in GR with $\gamma(t_1)=x$, $\gamma(t_2)=y$, and $\gamma(t)\in B$, for all $t\in[t_1,t_2]$. See Fig. 1.

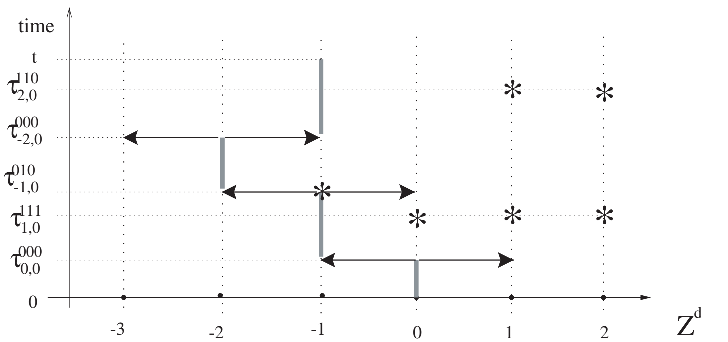

Fig. 1. The graphical representation GR. $(0,0)$ is connected to $(-1,t)$ by an open path. The trace is $(0,0)$, $(0,\tau_{0,0}^{000})$, $(-1,\tau_{0,0}^{000})$, $(-1,\tau_{-1,0}^{010})$, $(-2,\tau_{-1,0}^{010})$, $(-2,\tau_{-2,0}^{000})$, $(-1,\tau_{-2,0}^{000})$ $(-1,t)$.

For any finite $A \subseteq B \subset \mathbb{Z}$ and $t, s \geqslant 0$ we denote by $\xi_B^{(A,t)}(s)$ the random subset of vertices $x \in B$ such that there exists $y = y(x) \in A$, and $(y, t) \rightsquigarrow (x, t+s)$ in $GR|_B$.

The definition of $\xi_B^{(A,t)}(s)$ via the existence of certain open paths implies a number of useful properties. The first is monotonicity:

$$\xi_B^{(A,t)}(s) \subseteq \xi_D^{(C,t)}(s), \qquad s \geqslant 0 \qquad \text{if} \quad A \subseteq C, \quad B \subseteq D, \tag{2.1}$$

The second is the semigroup property:

$$\begin{aligned} &\text{(a)} \quad \xi_B^{(\{\varnothing\},0)}(t) = \{\varnothing\}, \qquad t \geqslant 0, \\ &\text{(b)} \quad \xi_B^{(\xi_B^{(A,t)}(s_1),\, t+s_1)}(s_2 - s_1) = \xi_B^{(A,t)}(s_2), \qquad \text{if } A \subseteq B, \quad 0 \leqslant s_1 \leqslant s_2. \end{aligned} \tag{2.2}$$

The monotonicity property (2.1) allows us to define the process $\xi_B^{(A,t)}(s)$ for any $A \subseteq B \subseteq \mathbb{Z}$:

$$\xi_B^{(A,t)}(s) = \lim_{\substack{B' \uparrow B,\\ B' \text{ finite}}} \xi_{B'}^{(A \cap B', t)}(s). \tag{2.3}$$

Note that due to the monotonicity property (2.1), the limit at the r.h.s. is independent of the sequence $B' \uparrow \mathbb{Z}$. The process $\xi(t)$ is now defined as follows:

$$\xi(t) = \xi_{\mathbb{Z}}^{(\mathbb{Z},0)}(t), \qquad t \geqslant 0. \tag{2.4}$$

We now extract the BS process $\eta_N(n)$ from the graphical representation GR as to have $\eta_N(n)$ and $\xi(t)$ defined on the same probability space.

The N vertices are labeled by $\Lambda(N) = \{-N'-1, \ldots, N''+1\}$, where $N' + 3 + N'' = N$, $N'' - 1 \leqslant N' \leqslant N''$, and the observation site is labeled by 0. We define $l(i)$ and $r(i)$ to be the left and right neighbours of i, respectively, with appropriate boundary conditions:

$$l(i) = \begin{cases} i-1, & i \in [-N', N''+1], \\ N''+1, & i = -N'-1, \end{cases}$$

$$r(i) = \begin{cases} i+1, & i \in [-N'-1, N''], \\ -N'-1, & i = N''+1. \end{cases}$$

A state of the BS process is determined by the subset of the vertices in state 0. Thus the state space $\mathcal{S}_N$ of the BS process consists of the all subsets of $\Lambda(N)$. If we denote the state of a site i in a configuration $\eta \in \mathcal{S}_N$ by η_i, then we have an identity

$$\eta_i = \mathbf{1}\{i \notin \eta\}. \tag{2.5}$$

This identity is natural because the 0's play the "active" role in the dynamics of BS process. It is possibly also slighty inconvenient for mathematicians, who are used to work with subsets of sites in state 1, (like in the contact process or in the 1-dim sendpile model, ect.). Indeed, for those processes we would have $\eta_i = \mathbf{1}\{i \in \eta\}$. Nevertherless, we will not reverse the roles of 1's and 0's, and will work with (2.5), because of the conventional definition of the BS-model.

We will now extract from GR two independent sequences of random variables $U = (U_j)$, $V = (V_j)$, and then we will define the BS process in terms of those sequences. Let $\Pi(N)$ denote the superposition of all the Poisson processes associated to the vertices in $\Lambda(N)$, i.e., with abuse of notation,

$$\Pi(N) = \bigcup_{k \in \Lambda(N)} (\Pi_k^{000} \cup \Pi_k^{001} \cup \cdots \cup \Pi_k^{110} \cup \Pi_k^{111}).$$

Then $\Pi(N)$ is a Poisson process on $\mathbb{R}$ with intensity $N/(1-q^3)$. Denote by

$$\tau(N) = (\tau_1, \tau_2, \ldots) \tag{2.6}$$

the arrivals of $\Pi(N)$ after time zero. For every $j \in \mathbb{N}$ there exists, with probability one, a unique $U_j \in \Lambda(N)$ and $V_j \in \{0, 1\}^3$ such that τ_j is the arrival of $\Pi_{U_j}^{V_j}$. It is clear that U, V and $\tau(N)$ are independent and each consists of i.i.d. random variables. Note that U_j, $j \in \mathbb{N}$ is uniformly distributed, that is, $P(U_j = i) = 1/N, i \in \Lambda(N)$. The sequence U will be used as a sequence of random vertices-canditates for the update procedure. The distribution of V_j is simply the joint distribution of three independent Bernoulli random variables, taking value 0 with probability q and 1 with probability $1-q$. The sequence V will be used to determine the states of the vertices after the updates.

We will define $\eta_N(n)$ inductively via the (random) increasing sequence $(j_n) \subset \mathbb{N}$. Let $j_0 = 0$, $\eta_N(0) = \varnothing$. Let $n \in \mathbb{N}$, and suppose that j_{n-1} and $\eta_N(n-1)$ are already defined. If $\eta_N(n-1) = \varnothing$, then $j_n = j_{n-1} + N + 1$, $\eta_N(n) = \{U_{j_n}\}$, i.e., we skip N elements of the sequences U and V, and then restart our process from the site U_{j_n}. The reason to skip N elements is that we want to have the following property: the more particles in state 1 we have in $\eta_N(n-1)$, the more elements of U and V, in mean, we skip to define $\eta_N(n)$. We will use this property later, in the proof of Lemma 2.1. If $\eta_N(n-1) \neq \varnothing$, we wait until we choose a vertex in state 0:

$$j_n = \min\{j > j_{n-1} \mid U_j \in \eta_N(n-1)\},$$

and change the state of site U_{j_n} and its neighbours $l(U_{j_n})$ and $r(U_{j_n})$ according the value of $V_{j_n} = (\sigma_1, \sigma_2, \sigma_3)$, i.e.,

$$\eta_N(n)_i = \begin{cases} \sigma_1, & i = l(U_{j_n}), \\ \sigma_2, & i = U_{j_n}, \\ \sigma_3, & i = r(U_{j_n}), \\ \eta_N(n-1)_i, & \text{otherwise.} \end{cases}$$

This finishes the construction of the process $\eta_N(n)$.

We will now introduce an "intermediate" continuous time process $\xi_N^R(t)$:

$$\xi_N^R(t) = \begin{cases} \eta_N(0), & t \in [0, \tau_1), \\ \eta_N(n(j)), & t \in [\tau_j, \tau_{j+1}), \quad j \geqslant 1, \end{cases}$$

where $(n(j))$ is defined as

$$n(j) = \max\{n \in \mathbb{N} : j_n \leqslant j\}.$$

It is clear that $\xi_N^R(t)$ is a continuous time Markov chain on $\mathcal{S}_N$. Observe that the processes ξ_N^R and η_N are related via a random time change. If there are many 1's around, then we typically skip more steps, so 1's tend to be preserved in ξ_N^R. This intuition is articulated in the following lemma.

Lemma 2.1. We have

$$\lim_{n \to \infty} P(\eta_N(n)_0 = 1) \leqslant \lim_{t \to \infty} P(\xi_N^R(t)_0 = 1). \tag{2.7}$$

Proof. We prove (2.7) in two steps:

$$\begin{aligned} &\text{(a)} \quad \lim_{n \to \infty} P(\eta_N(n)_0 = 1) \leqslant \lim_{j \to \infty} P(\eta_N(n(j))_0 = 1), \\ &\text{(b)} \quad \lim_{j \to \infty} P(\eta_N(n(j))_0 = 1) = \lim_{t \to \infty} P(\xi_N^R(t)_0 = 1). \end{aligned} \tag{2.8}$$

We prove (b) first. We write

$$P(\xi_N^R(t)_0 = 1) = \sum_{j=0}^{\infty} P(\eta_N(n(j))_0 = 1, \text{ and } \tau_j \leqslant t < \tau_{j+1}).$$

The random variable $\eta_N(n(j))_0$ is independent of τ_j and τ_{j+1}. Hence

$$|P(\xi_N^R(t)_0 = 1) - \lim_{j\to\infty} P(\eta_N(n(j))_0 = 1)|$$

$$= \left| \sum_{j=0}^{\infty} \{P(\eta_N(n(j))_0 = 1)\, P(\tau_j \leqslant t < \tau_{j+1})\} - \lim_{j\to\infty} P(\eta_N(n(j))_0 = 1) \right|$$

$$\leqslant \sum_{j=0}^{\infty} |P(\eta_N(n(j))_0 = 1) - \lim_{j\to\infty} P(\eta_N(n(j))_0 = 1)|$$

$$\times P(\tau_j \leqslant t < \tau_{j+1}) \to 0, \qquad \text{as} \quad t \to \infty,$$

becauce $\tau_j \to \infty$, as $j \to \infty$ in probability. This proves (b).

For (a), we write

$$p_k = \lim_{n\to\infty} P\left(\sum_{i \in \Lambda(N)} \eta_N(n)_i = k \right),$$

and

$$q_k = \lim_{j\to\infty} P\left(\sum_{i \in \Lambda(N)} \eta_N(n(j))_i = k \right).$$

It is then clear that

$$\lim_{n\to\infty} P(\eta_N(n)_0 = 1) = \sum_{k=0}^{N} \frac{k p_k}{N}$$

and

$$\lim_{j\to\infty} P(\eta_N(n(j))_0 = 1) = \sum_{k=0}^{N} \frac{k q_k}{N}.$$

Now observe that when there are $k < N$ vertices with fitness 1, the number of trials before we select a vertex with fitness 0 has a geometric distribution with parameter $(N-k)/k$, hence the expected number of trials is equal to $N/(N-k)$. It follows that for $k < N$,

$$q_k = \frac{\frac{N}{N-k}\, p_k}{\sum_{l=0}^{N-1} \frac{N}{N-l}\, p_l + (N+1)\, p_N}$$

and similarly

$$q_N = \frac{(N+1)\, p_N}{\sum_{l=0}^{N-1} \frac{N}{N-l}\, p_l + (N+1)\, p_N}.$$

We write $a_k = k/N$, $b_k = N/(N-k)$ for $k < N$, and $b_N = N+1$. Then

$$\begin{aligned} P(\eta_N(n(j))_0 = 1) &= \sum_{k=0}^{N} \frac{kq_k}{N} \\ &= \frac{\sum_{k=0}^{N-1} \frac{k}{N} \frac{N}{N-k}\, p_k + (N+1)\, p_N}{\sum_{l=0}^{N} b_l\, p_l} \\ &= \frac{\sum_{k=0}^{N} a_k b_k\, p_k}{\sum_{l=0}^{N} b_l\, p_l}. \end{aligned}$$

Now observe that for any probability vector $p_0,..., p_N$, and non-decreasing sequences $0 \leqslant a_0 \leqslant \cdots \leqslant a_N$ and $0 \leqslant b_0 \leqslant \cdots \leqslant b_N$, we have

$$\left(\sum_{k=0}^{N} a_k\, p_k \right) \left(\sum_{k=0}^{N} b_k\, p_k \right) \leqslant \sum_{k=0}^{N} a_k b_k\, p_k,$$

which can be proved by induction. Applying this general fact to the a_k's, b_k's and p_k's above, we find that the last quotient is bounded below by $\sum_{k=0}^{N} a_k\, p_k$ which is just $P(\eta_N(n)_0 = 1)$. (Note that here we have used the fact that we skip N choices if all vertices have fitness 1: this makes the sequence (b_k) increasing.) This proves (a). ∎

The next step in the proof of Theorem 1.1 is the following lemma, where we relate the process $\xi_N^R(t)$ to the graphical representation GR. To simplify notations further we will write GR_N instead of $GR|_{[-N', N'']}$.

Lemma 2.2. For any $t_1, t_2 \geqslant 0$, $x, y \in [-N', N'']$, if $\xi_N^R(t_1)_x = 0$ and $(x, t_1) \rightsquigarrow (y, t_2)$ in GR_N, then $\xi_N^R(t_2)_y = 0$.

Proof. Let $(x, t_1) \rightsquigarrow (y, t_2)$ in GR_N, with trace $(x_0, s_0),..., (x_n, s_n)$, i.e., every pair (x_i, s_i), (x_{i+1}, s_{i+1}) is connected either by a time segment or by an arrow. The statement will follow by induction, if we prove that $x_i \in \xi_N^R(s_i)$ implies $x_{i+1} \in \xi_N^R(s_{i+1})$, for every $0 \leqslant i \leqslant n-1$. Let $x_i \in \xi_N^R(s_i)$. If (x_i, s_i) is connected to (x_{i+1}, s_{i+1}) by a time segment, then there are no symbols "$*$" on (x_i, t), $t \in [s_i, s_{i+1})$. Hence, there are no arrivals at the time interval $[s_i, s_{i+1})$ at $\Pi_{x_i}^{\sigma_1, 1, \sigma_3}$, $\Pi_{x_i-1}^{\sigma_1, \sigma_2, 1}$ and $\Pi_{x_i+1}^{1, \sigma_2, \sigma_3}$, $(\sigma_1, \sigma_2, \sigma_3) \in \{0, 1\}^3$. Thus, $x_{i+1} = x_i \in \xi_N^R(s_{i+1})$, because only the above arrivals can delete x_i from $\xi_N^R(s_{i+1})$. If $x_{i+1} = x_i + 1$ and (x_i, s_i) is connected to (x_{i+1}, s_{i+1}) by an arrow, then there is an arrival at $\Pi_{x_i}^{\sigma_1, \sigma_2, 0}$, for some $\sigma_1, \sigma_2 \in \{0, 1\}$ at time $s_i = s_{i+1}$, and hence, $x_{i+1} \in \xi_N^R(s_{i+1})$. Similary, if $x_{i+1} = x_i - 1$ and (x_i, s_i) is connected to (x_{i+1}, s_{i+1}) by an arrow, then there is an arrival at $\Pi_{x_i}^{0, \sigma_2, \sigma_3}$, for some $\sigma_2, \sigma_3 \in \{0, 1\}$ at time $s_i = s_{i+1}$, and again, $x_{i+1} \in \xi_N^R(s_{i+1})$. ▮

The last two lemmas imply that we have reduced the problem to GR. Therefore, in the next section, we work in this graphical representation.

3. PROOF OF THEOREM 1.1

To simplify our notation $\xi_B^{(A, t)}(s)$, we will skip the upper index (A, t), if $A = B$ and $t = 0$. We will also skip the lower index B, if $B = \mathbb{Z}$. So for example, we will write $\xi^{(A, t)}(s)$ instead of $\xi_{\mathbb{Z}}^{(A, t)}(s)$, $\xi_{[x, \infty)}(t)$ instead of $\xi_{[x, \infty)}^{([x, \infty), 0)}(t)$, and $\xi_{(-\infty, x]}(t)$ instead of $\xi_{(-\infty, x]}^{((-\infty, x], 0)}(t)$. The idea to couple by a graphical representation the processes with various lower indices is taken from ref. 11.

Note that for any $t \geqslant 0$, with probability one, $\xi_{[-N', \infty)}(t) \neq \emptyset$ and $\xi_{(-\infty, N'']}(t) \neq \emptyset$. Thus we can define $l_{-N'}(t)$ and $r_{N''}(t)$ as the leftmost and rightmost 0's of the processes $\xi_{[-N', \infty)}(t)$ and $\xi_{(-\infty, N'']}(t)$, respectively. The following lemma was inspired by inequality (5.2) in ref. 11.

Lemma 3.1.

$$\text{(a)} \quad \xi_{[-N', \infty)}(t) \supseteq [l_{-N'}(t), \infty) \cap \xi(t), \qquad t \geqslant 0,$$

$$\text{(b)} \quad \xi_{(-\infty, N'']}(t) \supseteq (-\infty, r_{N''}(t)] \cap \xi(t), \qquad t \geqslant 0.$$

Proof. Let $y \in [l_{-N'}(t), \infty) \cap \xi(t)$. Then there exists $x_1 \in \mathbb{Z}$ and an open path $\gamma_1(s)|_0^t$ in GR such that $(x_1, 0)$ is connected to (y, t) by $\gamma_1(s)|_0^t$. Since $l_{-N'}(t) \in \xi_{[-N', \infty)}(t)$ there exists $x_2 \in \mathbb{Z} \cap [-N', \infty)$ and an open path $\gamma_2(s)|_0^t$, laying completely within $[-N', \infty) \times \mathbb{R}$ such that $(x_2, 0)$ is connected to $(l_{-N'}(t), t)$ by $\gamma_2(s)|_0^t$. Let s^* be defined as

$$s^* = \inf\{s \geqslant 0 : \gamma_2(s) \leqslant \gamma_1(s)\}.$$

Note that by definition any open path is a *cadlag* function of time, thus $\gamma_2(s^*) \leqslant \gamma_1(s^*)$. If $s^* = 0$ then the open path $\gamma_1(s)|_0^t$ lays completely within $[-N', \infty)$. If $s^* > 0$, we define the open path $\gamma_3(s)|_0^t$ as

$$\gamma_3(s) = \begin{cases} \gamma_2(s), & s \in [0, s^*), \\ \gamma_1(s), & s \in [s^*, t], \end{cases}$$

see Fig. 2. The open path $\gamma_3(s)|_0^t$ has endpoint (y, t) and lays completely within $[-N', \infty) \times \mathbb{R}$. Thus $(y, t) \in \xi_{[-N', \infty)}(t)$.

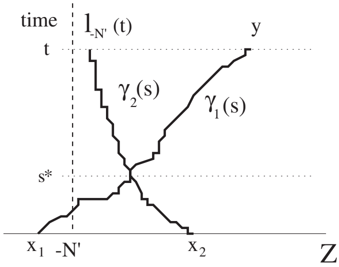

Fig. 2. The open path connecting $(x_2, 0)$ to $(\gamma_2(s^*), s^*)$ to (y, t) lays completely within $[-N', \infty) \times \mathbb{R}$.

The statement in (b) can be proved in a similar way. ∎

Define $[l, r]_N(t)$ as

$$[l, r]_N(t) = \xi_{[-N', \infty)}(t) \cap \xi_{(-\infty, N'']}(t), \qquad t \geqslant 0.$$

Lemma 3.2. If $[l, r]_N(t) \neq \emptyset$, for all $t \in [t_1, t_2]$, then for any $y \in [l, r]_N\, (t_2)$ there exists an open path $\gamma(t)|_{t_1}^{t_2}$ with $\gamma(t_2) = y$ and $\gamma(t) \in [l, r]_N\, (t)$, $t \in [t_1, t_2]$.

Proof. Let A' be the subset of $t^* \in [t_1, t_2]$ which have that for all $y \in [l, r]_N(t^*)$, there exists an open path $\gamma(s)|_{t_1}^{t^*}$ with $\gamma(t^*) = y$ and $\gamma(t) \in [l, r]_N(t)$, $t \in [t_1, t^*]$. If $[l, r]_N(t) \neq \emptyset$, for $t \in [t_1, t_2]$ then $l_{-N'}(t), r_N(t) \in [-N', N'']$, for all $t \in [t_1, t_2]$. Thus the event $t^* \in A'$ is determined only by the GR inside $[-N', N''] \times [t_1, t_2]$.

Suppose $t^* \in A'$ and τ is the first arrival of $\Pi(N)$ after time t^*. Then due to the definition of an open path $[t^*, \tau) \in A'$. We will prove that $\tau \in A'$. Then, by induction, $t_2 \in A'$, because there are only finitely many arrivals in $\Pi(N)$ at time interval $[t_1, t_2]$ and $t_1 \in A'$. Let $y \in [l, r]_N(\tau)$. Then there exist open paths $\gamma_1(t)|_{t_1}^{\tau} \in [-N', \infty)$ and $\gamma_2(t)|_{t_1}^{\tau} \in (-\infty, N'']$ with endpoint (y, τ). Since $l_{-N'}(t^*)$ is the leftmost point of $\xi_{[-N', \infty)}(t^*)$, we have $\gamma_1(t^*) \geqslant l_{-N'}(t^*)$. Similarly, $\gamma_2(t^*) \leqslant r_{N''}(t^*)$. The distance $|\gamma_1(t^*) - \gamma_2(t^*)| \leqslant 1$, because there is only one arrival of $\Pi(N)$ in the time interval $(t^*, \tau]$. It follows from the above that $\gamma_1(t^*) \in [l, r]_N(t^*)$ or $\gamma_2(t^*) \in [l, r]_N(t^*)$. Suppose $\gamma_1(t^*) \in [l, r]_N(t^*)$. Then $\gamma_1(t) \in [l, r]_N(t)$, $t \in [t^*, \tau]$. Since $t^* \in A'$ there exists an open path $\gamma(s)|_{t_1}^{t^*}$ with $\gamma(t^*) = \gamma_1(t^*)$ and $\gamma(t) \in [l, r]_N(t)$, $t \in [t_1, t^*]$. Hence, the path $\gamma_3(t)|_{t_1}^{\tau}$, defined as

$$\gamma_3(t) = \begin{cases} \gamma(t), & t \in [t_1, t^*), \\ \gamma_1(t), & t \in [t^*, \tau]. \end{cases}$$

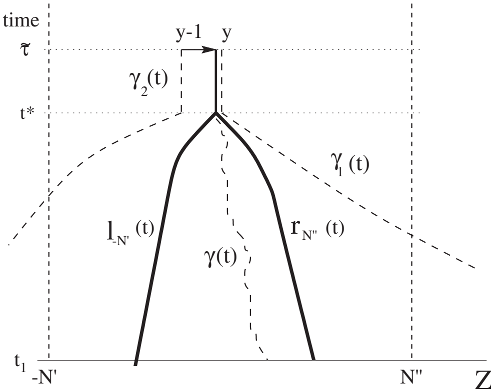

Fig. 3. At the realisation $[l, r]_N(\tau) = [l, r]_N(t^*) = \{y\}$. Since (y, t^*) is connected to (y, τ) by $\gamma(t)|_{t^*}^{\tau}$ and $t^* \in A'$, there exists an open path with endpoint (y, τ), laying completely within $[l, r]_N(t)$, $t \in [t_1, \tau]$.

has endpoint (y, τ), and satisfies $\gamma_3(t) \in [l, r]_N(t)$, $t \in [t_1, \tau]$. See Fig. 3 for an illustration. Thus, $\tau \in A'$. The case $\gamma_2(t^*) \in [l, r]_N(t^*)$ can be done similary.

We need a number of results which are very similar to the corresponding results for the standard contact process. Therefore, we omit the proofs of the following three lemmas. Their proofs are modifications of the proofs of Theorem 3.19, Theorem 3.21 and Corollary 3.22 in ref. 12. Note that in these three lemmas, we require q to be close enough to one. It is at this point where the rescaling of time by a factor $1/(1-q^3)$ comes in. With this rescaling of time, the intensity of arrows gets large when q gets close to one. The (omitted) proofs of the three lemmas involve comparison with oriented percolation, and in order to make sure that the appropriate oriented percolation model percolates, we need a high intensity of arrows.

Lemma 3.3. If q is close enough to one, then there exists $v(q) > 0$ such that for any $t > 0$

$$P(\xi(t)_0 = 0) > v(q). \tag{3.1}$$

Lemma 3.4. Let $x \in \mathbb{Z}$ and let $l_x(t)$ be the leftmost zero of $\xi_{[x,\infty)}(t)$, and $r_x(t)$ be the rightmost zero of $\xi_{(-\infty,x]}(t)$. If q is close enough to one, then there exist $c_1(q), c_2(q) > 0$, depending only on q, such that for any $m \in \mathbb{N}$ and $t > 5m^2$

$$\begin{aligned} &P(l_x(s) > x + m, \quad \text{for some} \quad s \in [t - 5m^2, t]) < c_1(q)\, e^{-c_2(q)\, m}, \\ &P(r_x(s) < x - m, \quad \text{for some} \quad s \in [t - 5m^2, t]) < c_1(q)\, e^{-c_2(q)\, m}. \end{aligned} \tag{3.2}$$

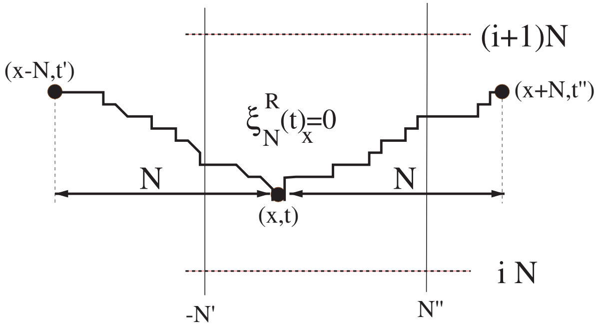

Fig. 4. The ith level is normalising.

Lemma 3.5. For q close enough to one there exists $c(q)>0$, depending only on q such that

$$P(\exists t' \in [0, N/2] : (0, 0) \rightsquigarrow (N, t') \text{ in } GR_{[0, \infty)}) > c(q).$$

Now we partition the time axis into intervals of length N, and we call the interval $[iN, (i+1) N)$ the ith *level*.

Definition 3.6. We call level i normalising if there exist $t, t', t'' \in [iN, (i+1) N)$ and $x \in [-N', N'']$ such that $\xi_N^R(t)_x = 0$, $(x, t) \rightsquigarrow (x-N, t')$ in $GR_{(-\infty, x]}$ and $(x, t) \rightsquigarrow (x+N, t'')$ in $GR_{[x, \infty)}$.

See Fig. 4 for an illustration of this definition.

We will use normalising levels to connect different open paths in GR. For this to work, we have to make sure that there are enough normalising levels. This is the content of the following key lemma.

Lemma 3.7. For q close enough to one, N large enough, and $T > N^2 + N$, there exists $c(q) > 0$, depending only on q, such that the probability to find no normalising level among levels $\lfloor T/N \rfloor - N, \dots, \lfloor T/N \rfloor$ is at most $e^{-c(q) N}$.

Proof. The events

$$\{i\text{th level is normalising}\}, \qquad i \in \mathbb{N}$$

are not independent. But we will construct independent events which guarantee that certain levels are normalising. To do this carefully, let $\mathscr{F}_i(N)$ be the σ-algebra generated by the restriction of GR to $\mathbb{Z} \times [0, iN+N)$. Let

$$t_i^* = \min\{iN + N/2, \inf\{t \geqslant iN : \xi_N^R(t) \cap [-N', N''] \neq \varnothing\}\},$$

and

$$x_i^* = \begin{cases} \min\{x \in [-N', N''] : \xi_N^R(t_i^*)_x = 0\}, & \text{if} \quad t_i^* < iN + N/2, \\ 0, & \text{if} \quad t_i^* = iN + N/2. \end{cases}$$

Let $A_i(N)$ be the event that $(x_i^*, t_i^*) \rightsquigarrow (x_i^* + N, t')$ in $GR_{[x_i^*, \infty)}$, and $(x_i^*, t_i^*) \rightsquigarrow (x_i^* - N, t'')$ in $GR_{(-\infty, x_i^*]}$, for some $t', t'' \in [t_i^*, t_i^* + N/2)$. Note that $A_i(N) \cap \{t_i^* < iN + N/2\}$ implies that the ith level is normalising. Since t_i^* is a stopping time and because of Lemma 3.5, there exists $p_{q,A} > 0$, depending only on q, such that

$$P(A_i(N)) \geqslant p_{q,A}, \tag{3.3}$$

uniformly in N. Observe that $A_i(N)$ is $\mathscr{F}_i(N)$-measurable, and is independent of $\mathscr{F}_{i-1}(N)$.

We are now going to make sure that $\{t_i^* < iN + N/2\}$ occurs often enough. Let s_i^* be defined as the smallest element s of the ith level for which one of the following three conditions is satisfied:

(1) $\xi_N^R(s) = \varnothing$;

(2) $\xi_N^R(s) \cap [-N', N''] \neq \varnothing$;

(3) $s = iN + N/4$.

Note that s_i^* is a stopping time (with respect to the natural filtration) and that $s_i^* \leqslant iN + N/4$. We will now give a condition in terms of GR within the ith level which ensures that $s_i^* < iN + N/4$.

If $\xi_N^R(iN) = \varnothing$ or $\xi_N^R(iN) \cap [-N', N''] \neq \varnothing$, then $s_i^* = iN$. The remaining cases are those where $\xi_N^R(i)$ is not empty and a subset of $\{-N'-1, N''+1\}$. We now *pretend* that $\xi_N^R(iN)$ is not empty and a subset of $\{-N'-1, N''+1\}$, giving three possible situations, namely $\xi_N^R(iN) = \{-N'-1\}$, $\xi_N^R(iN) = \{N''+1\}$ or $\xi_N^R(iN) = \{-N'-1, N''+1\}$. In each of these situations, we compute

$$\inf\{s \geqslant 0 : \xi_N^R(iN+s) = \varnothing \text{ or } \xi_N^R(iN+s) \cap [-N', N''] \neq \varnothing\},$$

which we denote by S^1, S^2 and S^3 respectively. We denote the maximum of these three numbers by S_i^*:

$$S_i^* = \max\{S^1, S^2, S^3\},$$

and define the event $B_i(N)$ as

$$B_i(N) = \{S_i^* < N/4\}.$$

Note that $B_i(N)$ is measurable with respect to the σ-algebra generated by GR restricted to the ith level, and therefore the $B_i(N)$'s are mutually independent for different i's. Also observe that occurrence of $B_i(N)$ implies that $s_i^* < N/4$, and that

$$\lim_{N \to \infty} P(B_i(N)) = 1. \tag{3.4}$$

Next, define $C_i(N)$ as the event that the $(N+1)$-th arrival of $\Pi(N)$ after time s_i^* takes place before time $s_i^* + N/4$. Since $\Pi(N)$ has intensity of order N, the number of arrivals of $\Pi(N)$ in a time interval of length $N/4$ has a Poisson distribution with mean of order N^2. This implies that

$$\lim_{N \to \infty} P(C_i(N)) = 1. \tag{3.5}$$

Finally, $D_i(N)$ is defined as the event that the first arrival of $\Pi(N)$ after time s_i^* is at a vertex in $[-N', N'']$, or that there is no such arrival during the time interval $[s_i^*, s_i^* + N/4)$. It is clear that

$$\lim_{N \to \infty} P(D_i(N)) = 1. \tag{3.6}$$

The events $C_i(N)$ and $D_i(N)$ are $\mathscr{F}_i(N)$-measurable, and independent of $\mathscr{F}_{i-1}(N)$. Now we have that

$$\{B_i(N) \cap C_i(N) \cap D_i(N)\} \subseteq \{t_i^* < iN + N/2\},$$

and therefore

$$\{A_i(N) \cap B_i(N) \cap C_i(N) \cap D_i(N)\} \subseteq \{i\text{th level is normalising}\}.$$

Also, for N large enough, we have, according to (3.3), (3.4), (3.5) and (3.6) that

$$P(A_i(N) \cap B_i(N) \cap C_i(N) \cap D_i(N)) > c_1(q),$$

for some $c_1(q) > 0$, uniformly in N. We may now write, using the independence of all events

$$\begin{aligned}
&P(\text{none of the levels } \lfloor T/N \rfloor - N, \ldots, \lfloor T/N \rfloor \text{ are normalising}) \\
&\leqslant P\left(\bigcap_{i=\lfloor T/N \rfloor - N}^{\lfloor T/N \rfloor} (A_i(N) \cap B_i(N) \cap C_i(N) \cap D_i(N))^c \right) \\
&= \prod_{i=\lfloor T/N \rfloor - N}^{\lfloor T/N \rfloor} P((A_i(N) \cap B_i(N) \cap C_i(N) \cap D_i(N))^c) \\
&\leqslant e^{-c(q)\,N},
\end{aligned}$$

for some $c(q) > 0$. ∎

Proof of Theorem 1.1. Due to the symmetry of BS-process we can work with $i = 0$. According to Lemma 2.1, it suffices to prove that for q close enough to one, there exists $c_q > 0$ such that for any N sufficiently large,

$$\lim_{t \to \infty} P(\xi_N^R(t)_0 = 0) \geqslant c_q.$$

For any $T > 0$ and N we have

$$P(\xi_N^R(T)_0 = 0) \geqslant P(\xi(T)_0 = 0) - P(\xi_N^R(T)_0 = 1, \xi(T)_0 = 0). \tag{3.7}$$

The first term in (3.7) is independent of N and positive, according to Lemma 3.3. Thus it is remains to prove that

$$P(\xi_N^R(T)_0 = 1, \xi(T)_0 = 0) \to 0, \qquad \text{as} \quad N \to \infty, \quad \text{uniformly in } T > T(N), \tag{3.8}$$

for some $T(N) < \infty$. Let $T > N^2 + N$. According to Lemma 3.4 and the stationarity of GR, for q close enough to one, there exists $c_1 = c_1(q)$, $c_2 = c_2(q) > 0$ depending only on q, such that

$$\begin{aligned}
&\text{(i)} \quad P(l_{-N'}(t) > 0, \qquad \text{for some} \quad t \in [T - N^2 - N, T]) < c_1 e^{-c_2 N}, \\
&\text{(ii)} \quad P(r_{N''}(t) \ < 0, \qquad \text{for some} \quad t \in [T - N^2 - N, T]) < c_1 e^{-c_2 N}.
\end{aligned} \tag{3.9}$$

It follows from (3.9) and Lemma 3.7 that

$$\begin{aligned}
&P(\xi_N^R(T)_0 = 1, \xi(T)_0 = 0) \\
&\leqslant P\left(\begin{array}{c} \xi_N^R(T)_0 = 1,\ \xi(T)_0 = 0, \\ l_{-N'}(t) \leqslant 0 \leqslant r_{N''}(t), \text{ for all } t \in [T - N^2 - N, T], \\ \exists i \in \{\lfloor T/N \rfloor - N, \ldots, \lfloor T/N \rfloor\} : i\text{th level is normalising} \end{array} \right) + c_1 e^{-c_2 N}
\end{aligned} \tag{3.10}$$

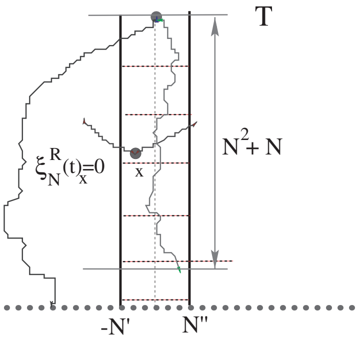

Fig. 5. The main idea of the proof. With probability close to one, $\{\xi(T)_0=0\}$ implies $\{\xi_N^R(T)_0=0\}$.

Consider the first term of (3.10). It follows from Lemma 3.1 that the event

$$\{l_{-N'}(T)\leqslant 0\leqslant r_{N''}(T),\ \xi(T)_0=0\}$$

implies that $\{0\}\in[l,r]_N(T)$. Then, by Lemma 3.2, the event

$$\left\{\begin{array}{c}\{0\}\in[l,r]_N(T),\\ l_{-N'}(t)\leqslant r_{N''}(t),\ \text{ for all } t\in[T-N^2-N,T],\end{array}\right\}$$

implies that there exists an open path $\gamma(t)|_{T-N^2-N}^{T}$ with endpoint $(0,T)$ and laying completely within $[-N',N'']$. If the ith level is normalising then, by definition, there exists $t,t',t''\in[iN,(i+1)N)$, $x\in[-N',N'']$ and open paths $\gamma_l(s)|_t^{t'}:(x,t)\rightsquigarrow(x-N,t')$ in $GR_{(-\infty,x]}$, $\gamma_r(s)|_t^{t''}:(x,t)\rightsquigarrow(x+N,t'')$ in $GR_{[x,\infty)}$. Either $\gamma_l(s)|_t^{t'}$ or $\gamma_r(s)|_t^{t''}$ intersects $\gamma(t)|_{T-N^2-N}^{T}$, see Fig. 5. Hence (x,t) is connected to $(0,T)$ by an open path in GR_N, and by Lemma 2.2 we have $\xi_N^R(T)_0=0$. Hence the first term at the right hand side of (3.10) equals zero and the second term gives us the theorem. ∎

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